tolerance fastener holes with and without fastener-load transfer.

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A Simplified Algorithm for Determining the Stability of Linear Systems

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Introduction

IN aeroelasticity, as well as in other nonconservative problems in the theory of elastic stability, there is often a need for information concerning the stability of small motions about an equilibrium state. Such analyses can generally be reduced to standard eigenvalue problems with a host of well-known methods of solution. The eigenvalues may provide too much information, however. The analyst may only want to know whether the motion is stable. To determine the stability for a system of n equations of order r by classical methods first requires extracting $N=nr$ eigenvalues and then testing each real part to see if it is positive (unstable) or negative (stable). A scheme that requires calculation of only one number to be tested would be clearly superior, especially if the computation time was less than that of extracting eigenvalues. Such a scheme has been developed and is the subject of this paper.

Analysis

We assume that the equations of motion may be reduced to the eigenvalue problem

$$Px=\lambda x$$

where P is the stability matrix, x is the vector of system generalized coordinates, and λ is the set of eigenvalues. The matrix P is real and of order $N\times N$. Extracting the eigenvalues is quite straightforward, but if the stability of the system is the extent of our concern then we should take a different approach.

We first should reduce P to an upper Hessenberg matrix H by so-called similarity transformations.[1] This step is available in many standard subroutine packages (e.g., Refs. 2 and 3).

Received June 29, 1976; revision received Nov. 23, 1976.
Index categories: Aeroelasticity and Hydroelasticity; Structural Dynamic Analysis.

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The form of H is

$$H=\begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,N-1} & h_{1,N} \\ k_2 & h_{2,2} & h_{2,3} & \cdots & h_{2,N-1} & h_{2,N} \\ 0 & k_3 & h_{3,3} & \cdots & h_{3,N-1} & h_{3,N} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & k_N & h_{N,N} \end{bmatrix}$$

The next step is to generate the characteristic polynomial. The method of Leverrier[4] is often used. The method used here[5] is more efficient than Leverrier's method, which requires about N times as many multiplications. If we denote by $p_r(\lambda)$ the characteristic polynomial of the square submatrix H_r of order r, then expanding $\det(H_r-\lambda I)$ in terms of its rth column yields

$$p_0(\lambda)=1$$

$$p_1(\lambda)=h_{1,1}-\lambda$$

$$p_r(\lambda)=(h_{r,r}-\lambda)p_{r-1}(\lambda)-h_{r-1,r}k_r p_{r-2}(\lambda)$$
$$+h_{r-2,r}k_r k_{r-1}p_{r-3}(\lambda)-\ldots$$
$$+(-1)^{r-1}h_{1,r}k_r k_{r-1}\ldots k_2 p_0(\lambda)$$

To derive the coefficients of $p_N(\lambda)$, we must store those of $p_r(\lambda)$ for $r=1, 2,\ldots,N-1$. The coefficients of $p_N(\lambda)$ are denoted by c_i, where $c_1=(-1)^N$ and

$$\sum_{i=1}^{N+1} c_i\lambda^{N-i+1}=0$$

A Fortran coding for this part of the numerical scheme may be obtained from the author.

The stability of the system may be determined by the Routh-Hurwitz criterion.[6] The conditions for instability are

$$c_{N+1}<0 \quad \text{(divergence)}$$

or

$$\Delta_{N-1}< \quad \text{(flutter)}$$

where

$$\Delta_{N-1}=\begin{vmatrix} c_2 & c_4 & c_6 & \cdots & c_{2N-2} \\ c_1 & c_3 & c_5 & \cdots & c_{2N-3} \\ 0 & c_2 & c_4 & \cdots & c_{2N-4} \\ 0 & c_1 & c_3 & \cdots & c_{2N-5} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & & & c_N \end{vmatrix}$$

and $c_i=0$ for $i>N+1$. Any convenient scheme for evaluating the determinant Δ_{N-1} may be used such as the LU decomposition. It would be faster, however, to take advantage of the form of Δ_{N-1} as discussed in Ref. 6. The computation of $c_i, i=1, 2,\ldots,N+1$ and of Δ_{N-1}, combined, requires fewer multiplications than calculating all the eigenvalues (even using the slower LU decomposition to evaluate Δ_{N-1} as is done in the numerical example below, and the dynamic stability (flutter) is known by simply checking the sign of one number. The static stability (divergence) can be more efficiently determined from a static criterion alone without introducing dynamic effects. A combination of this scheme and linear interpolation provides rapid convergence to a stability boundary indicated by a change in sign of Δ_{N-1}. This

operation has proven to be extremely helpful in plotting stability boundaries for systems in which the elements of P are functions of parameters that are being varied. The scheme has a tendency to break down for large matrices (25×25 or larger) due to the extremely large numbers generated in taking the determinant Δ_{N-1}. This can be partially alleviated by using double precision and by normalizing the original matrix P so that the numerical size of the elements c_i is decreased. The particular way in which P is normalized is not crucial so long as the eigenvalues are not affected except within a multiplicative constant.

Numerical Example

For an example of a dynamic system as described in the preceding, we assume that the equation of motion is of the following form.

$$I\ddot{x} + (\delta S + \epsilon I)\dot{x} + (K + aS)x = 0$$

where I is the $n \times n$ identity matrix and elements S and K are given

$$S_{i,j} = \begin{cases} 0, & i=j \\ +1, & i<j \\ -1, & i>j \end{cases} \Bigg\} i,j = 1,2,\ldots,n$$

$$K_{i,j} = \begin{cases} 0, & i \neq j \\ k^{2i-2}, & i=j \end{cases}$$

and where δ, ϵ, k and a are scalars reflecting the strength of each matrix. The stability of motions described by x is reflected by the eigenvalues of the $N \times N$ matrix P where

$$P = \begin{bmatrix} 0 & I \\ -(K+aS) & -(\delta S + \epsilon I) \end{bmatrix}$$

and $N = 2n$. In Fig. 1, we compare the relative CPU time for extracting the eigenvalues using the QR algirithm and for calculating both c_i for $i = 1,2,\ldots,N+1$ and Δ_{N-1}. As is evident, the calculation of Δ_{N-1} is from 15 to 30% faster for $N > 3$. The percentage improvement tends to decrease as N increases, but the actual difference between the CPU times (not shown) increases monotonically with N. In Fig. 2, we show $Re(\lambda_1)$, the real part of the eigenvalue associated with the lowest frequency and Δ_5 for $n = 3$, $\epsilon = 0.001$, $a = -0.5$, $k = 5$, and δ varying. The crossing of a stability boundary is evident in the change in sign of $Re(\lambda_1)$ and Δ_5. The two methods predict the same location for the stability boundary as indicated by their crossing of the horizontal axis at the same value of δ.

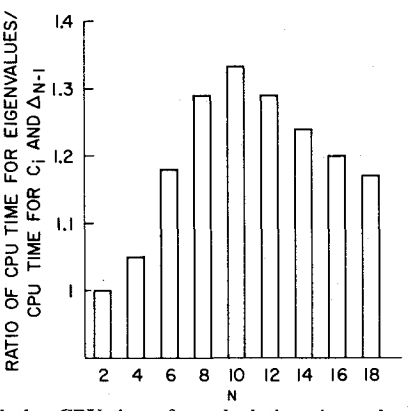

Fig. 1 Relative CPU times for calculating eigenvalues and stability information vs $N(N = 2n)$.

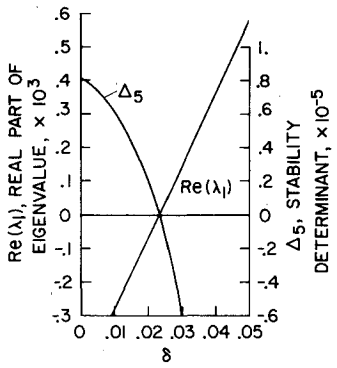

Fig. 2 $Re\ (\lambda_1)$ and Δ_5 vs δ for $a = -\frac{1}{2}$, $k = 5$, $\epsilon = 0.001$, $n = 3$.

Concluding Remarks

A simplified numerical scheme is presented which enables the analyst to determine the stability of a linear system by checking the sign of one number. The scheme has distinct advantages. Not only is it faster than extracting eigenvalues, but it also has a simpler application in determining stability. These advantages facilitate the generation of stability boundaries in parameter studies. It has a disadvantage of tending to break down for large matrices. This can be overcome partly by using double precision and by normalizing the matrix from which stability information is derived.

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Compressible Turbulent Boundary Layers with Injection

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Nomenclature

c_f = skin-friction coefficient
F = injection parameter $(\rho_w v_w / \rho_1 U_1)$ based on freestream conditions
m = injection parameter $(\rho_w v_w / \rho_o U_o)$ based on freestream conditions upstream of the pressure gradient

Introduction

As part of a general study[1-3] of compressible turbulent boundary layers with fluid injection at the surface, we

Received July 6, 1976; revision received Nov. 15, 1976.
Index category: Boundary Layers and Convective Heat Transfer—Turbulent.

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